Fall 24

MATH 5A - TEST 2 v1
(2.1ii-2.6, 2.8, ~~2.9~~)

100 points **NAME:_______________**

Show your work using clear presentation and explanation.
Correct notation must be used. Phones must be OFF and put away. No graphing calculators allowed.

When computing derivatives,

- *Use correct notation*
- *Label f, f' clearly*
- *Simplify* $f'(x)$:
  - *No complex fractions,*
  - *No negative exponents,*
  - *Combine fractions*

---

(1) For the function $y = f(x) = x^5$, circle T if the following notation is correct, F if incorrect.

(2 points each)

T F (a) $\frac{dy}{dx} = 5x^4$

T F (b) $\frac{d}{dx} = 5x^4$.

T F (c) $\frac{dy}{dx}(x^5) = 5x^4$

T F (d) $\frac{d}{dx}(x^5) = 5x^4$

T F (e) $dy = 5x^4 dx$ differential

---

(2) Find $f'(x)$. No partial credit (6 points)

a) $f(x) = \frac{7}{x^2} + \frac{2x^2}{5} = 7x^{-2} + \frac{2}{5}x^2$

$f'(x) = -14x^{-3} + \frac{4}{5}x$

$f'(x) = \frac{-14}{x^3} + \frac{4}{5}x$

$f'(x) = \frac{-70 + 4x^4}{5x^3}$

b) $f(x) = \sqrt{x}\left(2x^2 + \sqrt{x}\right)$ Easier simplify f first

$f(x) = 2x^{5/2} + x$

$f'(x) = 5x^{3/2} + 1$

see 2.3 video 2 for tips on problems like these

(3) Find $\frac{dy}{dx}$ No partial credit (8 points)

easier to reduce first

a). $y = \frac{5x^3}{x^3+2x^2} = \frac{5x}{x+2}$

quotient rule

$$\frac{dy}{dx} = \frac{(x+2)5 - 5x}{(x+2)^2} = \boxed{\frac{10}{(x+2)^2}}$$

Longer way

$$\frac{dy}{dx} = \frac{(x^3+2x^2)(15x^2) - 5x^3(3x^2+4x)}{(x^3+2x^2)^2}$$

$$= \frac{15x^5+30x^4-15x^5-20x^4}{(x^3+2x^2)^2}$$

$$= \frac{10x^4}{(x^3+2x^2)^2}$$

$$= \frac{10x^4}{x^4(x+2)^2}$$

$$= \frac{10}{(x+2)^2}$$

b) $y = 7x^2\sec(x)$

product rule

$$\boxed{\frac{dy}{dx} = 14x\sec(x) + 7x^2\sec(x)\tan(x)}$$

(4) Use differentials or linear approximation to approximate $\sqrt[4]{0.99}$ (10 points)

Let $f(x) = \sqrt[4]{x}$ so $f'(x) = \frac{1}{4x^{3/4}}$

$f(1) = 1$

$f'(1) = \frac{1}{4}$

Let $a = 1$ (# near 0.99 for which f(a) easy to find

Tangent line: $L(x) = f(a) + f'(a)(x-a)$

$= f(1) + f'(1)(x-1)$

$L(x) = 1 + \frac{1}{4}(x-1)$

$$\sqrt[4]{.99} = f(0.99) \approx L(0.99) = 1 + \frac{1}{4}(.99-1)$$

$= 1 + \frac{1}{4}(-.01)$

$= 1 + (.25)(-.01)$

$= 1 - .0025$

$= \boxed{.9975}$

In problems 5- 6 find $\frac{dy}{dx}$. Work carefully, very limited partial credit will be given. (8 pts each)

(5) $y = \frac{x^3 \cos x}{7-6x}$ quotient

(multiply out numerator + combine like terms!

$$\frac{dy}{dx} = \frac{(7-6x)\frac{d}{dx}(x^3\cos x) - x^3\cos x\,\frac{d}{dx}(7-6x)}{(7-6x)^2}$$

product

$$= \frac{(7-6x)(3x^2\cos x - x^3\sin x) - x^3\cos x(-6)}{(7-6x)^2}$$

$$= \frac{21x^2\cos x - 7x^3\sin x - 18x^3\cos x + 6x^4\sin x + 6x^3\cos x}{(7-6x)^2}$$

$$\boxed{= \frac{21x^2\cos x - 7x^3\sin x - 12x^3\cos x + 6x^4\sin x}{(7-6x)^2}}$$

5

(6) $y = \sin^4(8x^2) = (\sin(8x^2))^4$ "3 layers"

chain

chain

$$\frac{dy}{dx} = 4(\sin(8x^2))^3\,\frac{d}{dx}(\sin(8x^2))$$

$$= 4\sin^3(8x^2)\cos(8x^2)\,\frac{d}{dx}(8x^2)$$

$$= 4\sin^3(8x^2)\cos(8x^2)\cdot 16x$$

$$\boxed{\frac{dy}{dx} = 64x\sin^3(8x^2)\cos(8x^2)}$$

Similar to HW 2.5 #41

In problems 7- 8 find $f'(x)$. Work carefully, very <u>limited</u> partial credit will be given.
(8 pts each)

(7) $y = \dfrac{x}{\sqrt[3]{3x+1}} = x(3x+1)^{-1/3}$

$$\frac{dy}{dx} = \frac{d}{dx}(x)\,(3x+1)^{-1/3} + x\,\frac{d}{dx}(3x+1)^{-1/3}$$

chain

$$= (3x+1)^{-1/3} + x \cdot -\tfrac{1}{3}(3x+1)^{-4/3} \cdot 3$$

$$= (3x+1)^{-1/3} - x(3x+1)^{-4/3}$$

$$= (3x+1)^{-4/3}(3x+1-x)$$

$$= \boxed{\frac{2x+1}{(3x+1)^{4/3}}}$$

---

(8) $f(x) = \tan\left(\dfrac{x^2}{2x+1}\right)$

chain

$$\frac{dy}{dx} = \sec^2\left(\frac{x^2}{2x+1}\right)\frac{d}{dx}\left(\frac{x^2}{2x+1}\right)$$

quotient

$$= \sec^2\left(\frac{x^2}{2x+1}\right)\frac{(2x+1)2x - x^2(2)}{(2x+1)^2}$$

$$= \boxed{\sec^2\left(\frac{x^2}{2x+1}\right)\frac{2x^2+2x}{(2x+1)^2}}$$

see second sample test, #9

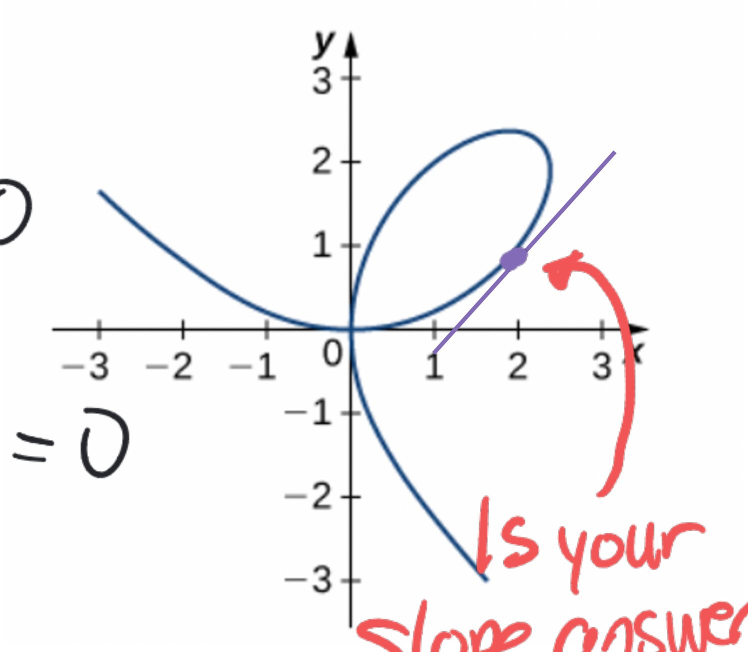
(9) Find the equation of the tangent line to the curve $2x^3 + 2y^2 - 9xy = 0$ at the point (2,1).
(Graph is provided to help you check whether your answer is reasonable if you want to)
(10 points)

Need $\frac{dy}{dx}$ - Use implicit differentiation

$$\frac{d}{dx}\left(2x^3 + 2y^2 - 9xy\right) = \frac{d}{dx}(0)$$

product

$$6x^2 + 4y\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0$$



Is your slope answer reasonable?

$$6x^2 + 4y\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$

$$6x^2 + 4y\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0$$

$$4y\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 6x^2$$

$$\frac{dy}{dx}(4y - 9x) = 9y - 6x^2$$

$$\boxed{\frac{dy}{dx} = \frac{9y - 6x^2}{4y - 9x}}$$

Need slope $\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{9-24}{4-18} = \frac{-15}{-14} = \frac{15}{14}$

Seems reasonable

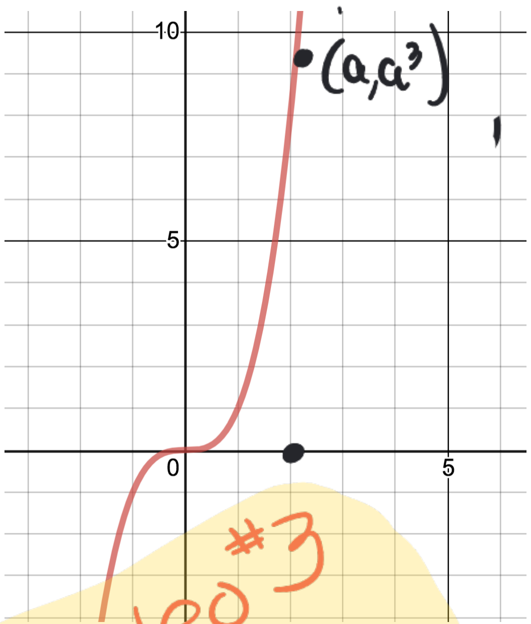
Line

$$y - 1 = \frac{15}{14}(x - 2)$$

OR $$y = \frac{15}{14}x - \frac{22}{7}$$

Similar to book's example 2 in 2.6

(10). (a). Find equation(s) of the tangent line(s) to $f(x) = x^3$ that contain(s) the point (2,0)
(b) Graph your line(s) on the given grid to see if your answer is reasonable. (14 points)



See video #3 from 2.3 – same problem

Note: (2,0) is not the point of tangency.

Let $(a, f(a)) = (a, a^3)$ be the unknown point of tangency.

Tangent line through (a, a^3)

$$y - f(a) = \underbrace{f'(a)}_{m}(x - a)$$

$$y = a^3 + 3a^2(x - a)$$

Line contains (2,0) also so

$$0 = a^3 + 3a^2(2 - a)$$

solve for a

$$0 = a^3 + 6a^2 - 3a^3$$

$$0 = 6a^2 - 2a^3$$

$$0 = 2a(3 - a)$$

$$a = 0, a = 3$$

$\underline{a = 0}$

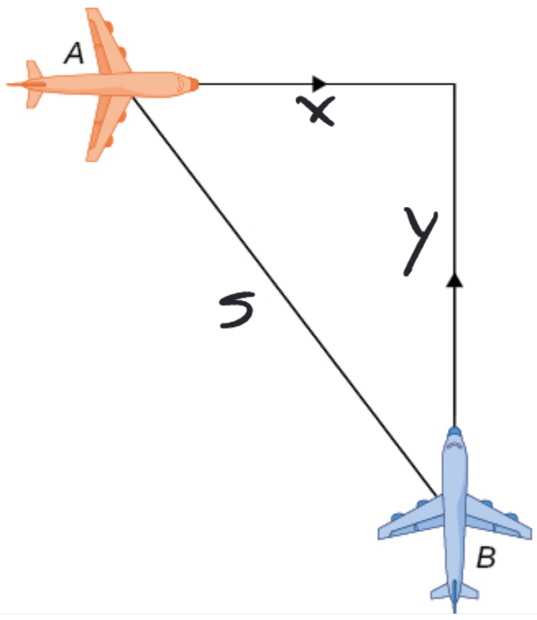
$\boxed{y = 0}$

$\underline{a = 3}$ $(3, 27)$ $m = f'(3) = 27$

$\boxed{y - 27 = 27(x - 3)}$

(11)

7. Two airplanes are flying in the air at the same height: airplane A is flying east at 250 mi/h and airplane B is flying north at 300 mi/h. If they are both heading to the same airport, located 30 miles east of airplane A and 40 miles north of airplane B, at what rate is the distance between the airplanes changing?



Know

$\frac{dx}{dt} = -250 \quad \frac{dy}{dt} = -300$

Want

$\left.\frac{ds}{dt}\right|_{x=30,\ y=40}$

$x^2 + y^2 = s^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$

(10 points)

$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{s}$$

$$\left.\frac{ds}{dt}\right|_{x=30,\ y=40} = \frac{30(-250)+40(-300)}{50}$$

$$= \frac{-19500}{50}$$

$$= \boxed{-390 \text{ mi/hr}}$$

See example 4 in sec 2.8 for similar problem